INTRINSIC VOLUMES AND LATTICE POINTS OF CROSSPOLYTOPES

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ABSTRACT. Hadwiger showed by computing the intrinsic volumes of a regular simplex that a rectangular simplex is a counterexample to Wills's conjecture for the relation between the lattice point enumerator and the intrinsic volumes in dimensions not less than 441.

Here we give formulae for the volumes of spherical polytopes related to the intrinsic volumes of the regular crosspolytope and of the rectangular simplex. This completes the determination of intrinsic volumes for regular polytopes. As a consequence we prove that Wills's conjecture is false even for centrally symmetric convex bodies in dimensions not less than 207.

1. Introduction

In convexity the intrinsic volumes play a central rle. They are most easily defined by the STEINER formula: We denote by $\mathcal{K}^d$ the family of all convex bodies — compact convex sets — in the d-dimensional Euclidean space E^d. Then the volume of the outer parallel body K_ρ at distance $\rho \geq 0$ of $K \in \mathcal{K}^d$ is a polynomial in ρ [BF]. In terms of the intrinsic volumes $V_i(K)$, $0 \leq i \leq d$, this polynomial is given by

$$V(K_\rho) = \sum_{i=0}^{d} V_i(K)\kappa_{d-i}\rho^{d-i},$$

where κ_i denotes the i-dimensional volume of the i-dimensional unit ball [Mc]. In particular we have that $V_d(K)$ is the volume of K and $V_{d-1}(K)$ is half of its surface area.

For polytopes $P \in \mathcal{K}^d$ we have a more explicit formula. To describe this formula let $\mathcal{F}_i(P)$ denote the set of all i-dimensional faces of P and for $F \in \mathcal{F}_i(P)$ let $c(F)$ denote the positive hull of all outward normals of supporting hyperplanes of F, embedded in the E^{d-i}. The external angle of a face F is denoted by $\gamma(F)$, that is the ratio of the spherical volume of $c(F) \cap S^{d-i-1}$ to the spherical volume of S^{d-i-1}, where S^{d-i-1} denotes the $(d-i)$-dimensional unit sphere. With this notation the intrinsic volumes of a polytope $P \in \mathcal{K}^d$ become [Mc]

$$V_i(P) = \sum_{F \in \mathcal{F}_i(P)} \gamma(F) V^i(F), \tag{1.1}$$

where $V^i(\cdot)$ denotes i-dimensional volume.

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Thus the computation of the V_i's involves the computation of the volumes of spherical polytopes, for which we have explicit formulae only in small dimensions (see e.g. [BöH]). In fact except for boxes, for which the problem is trivial as the normal cones are orthants, only the external angles for the regular simplex are known in high dimensions, as here the intersections $c(F) \cap S^{d-i-1}$ are regular spherical simplices, whose volumes were determined by Ruben [R], see also Hadwiger [H].

Here we give a formula for the $V_i(C^d)$, $i = 0, \dots, d$, where $C^d = \mathrm{conv}\{\pm e^i, 1 \leq i \leq d\}$ with e^i the i-th standard unit vector, denotes the regular d-dimensional crosspolytope. Thus the intrinsic volumes of all the regular polytopes are now determined, as the remaining cases in $d = 3, 4$ are trivial.

In section 3 we use the results to study crosspolytopes as examples for the relation between the lattice point enumerator $G(K) = \mathrm{card}(K \cap \mathbb{Z}^d)$ and the intrinsic volumes. For a general overview on this topic we refer to [GL, Chapter 2] or [BW]. A satisfactory lower bound for $G(K)$ in terms of the intrinsic volumes was given by the result of Bokowski, Hadwiger and Wills [BHW], who showed that

$$G(K) \geq V_d(K) - V_{d-1}(K),$$

and that this bound is best possible. In the other direction far less is known. Wills [W] gave constants λ_i, such that

$$G(K) \leq \sum_{i=0}^{d} \lambda_i V_i(K).$$

Further he conjectured $\lambda_i = 1$ for all i. In terms of Wills's functional $W(K) = \sum_{i=0}^{d} V_i(K)$ this becomes $G(K)/W(K) \leq 1$. The conjecture was proved in some special cases as e. g. for $d = 2$ by Nosarzewska [N] and for $d = 3$ by Overhagen [O]. Further it is easy to see that the conjecture cannot be improved as there is equality for all lattice boxes with axes parallel to the coordinate axes.

Rather surprisingly it was disproved by Hadwiger [H] by the following simplex: Let R^d be the rectangular simplex with vertices $0, e^1, \dots, e^d$. Then Hadwiger showed that $\sqrt{d}R^d$ is a counterexample to $G(K) \leq W(K)$ for $d \geq 441$. (With the same numerical calculations used by Hadwiger one can show that even $15 \cdot R^{410}$ is a counterexample.)

Here we study the behaviour of $G(aC^d)/W(aC^d)$, $a > 0$. We first show that in a certain sense crosspolytopes are better counterexamples than simplices (Theorem 3.1.). This shows specifically that Wills's conjecture remains wrong even if only bodies which are symmetrical with respect to the origin are considered. Finally, we use the results from section 2 to show that Wills's conjecture becomes wrong for $d \geq 207$, thus narrowing the gap for which it is still open.

2. Intrinsic volumes of regular crosspolytopes

Lemma 2.1. *Let F^i be an i-dimensional face of C^d, $0 \leq i < d$. The external angle of F^i is given by*

$$\gamma(F^i) = \frac{2^{d-i-1}}{\sqrt{\pi}^{d-i}} \int_0^\infty e^{-x^2} \left(\int_0^{x/\sqrt{i+1}} e^{-y^2}\, dy \right)^{d-i-1} dx.$$

Proof. Consider the i-dimensional face $F^i = \operatorname{conv}\{e^{d-i}, \dots, e^d\}$ of C^d. The 2^{d-i-1} outward normal vectors of the supporting hyperplanes of the facets containing F^i are given by $\{\sum_{k=1}^{d-i-1} j_k e^k + \sum_{k=d-i}^{d} e^k,\ j_k \in \{-1,1\}\}$. The normal cone $c(F^i)$ is the positive hull of these normal vectors, and it follows that (cf. [H])

$$\begin{aligned}\int_{c(F^i)} e^{-\|x\|^2} dx &= \gamma(F) \cdot V^{d-i-1}(S^{d-i-1}) \int_0^\infty e^{-r^2} r^{d-i-1} dr \\ &= \gamma(F^i) \cdot (d-i) \cdot \kappa_{d-i} \cdot \frac{\Gamma((d-i)/2)}{2} = \gamma(F^i) \cdot \pi^{(d-i)/2},\end{aligned} \tag{2.1}$$

where $V^{d-i-1}(S^{d-i-1})$ denotes the spherical volume of S^{d-i-1}. Now, let $U = \{x \in E^{d-i} \mid x_{d-i} \geq 0,\ |x_k| \leq x_{d-i},\ 1 \leq k \leq d-i-1\}$ and $f : U \to c(F^i)$ the linear and bijective map $f(x_1, \dots, x_{d-i}) = \sum_{k=1}^{d-i-1} x_k e^k + x_{d-i} \sum_{k=d-i}^{d} e^k$. From this we get

$$\begin{aligned}\int_{c(F^i)} e^{-\|x\|^2} dx &= \sqrt{i+1} \int_U e^{-\|f(x)\|^2} dx \\ &= \sqrt{i+1} \int_0^\infty \int_{-x_{d-i}}^{x_{d-i}} \cdots \int_{-x_{d-i}}^{x_{d-i}} e^{-(x_1^2 + \cdots + x_{d-i-1}^2 + (i+1)x_{d-i}^2)} dx_1 \dots dx_{d-i} \\ &= \sqrt{i+1} \int_0^\infty e^{-(i+1)x^2} \left(\int_{-x}^{x} e^{-y^2} dy \right)^{d-i-1} dx \\ &= 2^{d-i-1} \int_0^\infty e^{-x^2} \left(\int_0^{x/\sqrt{i+1}} e^{-y^2} dy \right)^{d-i-1} dx.\end{aligned}$$

Together with (2.1) we get the desired formula. $\square$

In particular we have $\gamma(F^0) = (2d)^{-1}$ and $\gamma(F^{d-1}) = 1/2$. Every i-dimensional face of C^d is a regular i-simplex with edge length $\sqrt{2}$, and hence the volume of such a face is $\sqrt{i+1}/i!$. Further, the number of i-faces of C^d is $2^{i+1}\binom{d}{i+1}$ [McS, pp. 80] and so we have by (1.1)

Theorem 2.1. *The intrinsic volumes of the regular crosspolytope $C^d \subset E^d$ are given by the following formulae:*

$$V_d(C^d) = \frac{2^d}{d!},$$

and for $0 \leq i \leq d-1$,

$$V_i(C^d) = 2^d \binom{d}{i+1} \frac{\sqrt{i+1}}{i! \sqrt{\pi}^{d-i}} \times \int_0^\infty e^{-x^2} \left(\int_0^{x/\sqrt{i+1}} e^{-y^2} dy \right)^{d-i-1} dx.$$

With the same methods the intrinsic volumes of the rectangular simplex R^d can be computed. We only have to observe that there are two kinds of i-faces. We omit the calculation but only state:

Theorem 2.2. *The intrinsic volumes of the rectangular simplex $R^d \subset E^d$ are given by the following formulae:*

$$V_d(R^d) = \frac{1}{d!},$$

and for $0 \leq i \leq d-1$,

$$V_i(R^d) = \binom{d}{i}\frac{1}{i!2^{d-i}} + \binom{d}{i+1}\frac{\sqrt{i+1}}{i!\sqrt{\pi}^{d-i}} \times \int_0^\infty e^{-x^2}\left(\int_{-\infty}^{x/\sqrt{i+1}} e^{-y^2}dy\right)^{d-i-1} dx.$$

Let us remark that the external angles of R^d can also be computed by applying Schläfli's recursive differential equation [Sch], but this is a more laborious method.

3. Crosspolytopes as examples in Wills's conjecture

The next theorem shows that regular crosspolytopes are better counterexamples to Wills's conjecture than rectangular simplices:

Theorem 3.1. *For every $a > 0$ and for every $d \in \mathbb{N}$ there is a $d_o \leq d$, such that*

$$\frac{G(aC^{d_0})}{W(aC^{d_0})} \geq \frac{G(aR^d)}{W(aR^d)}.$$

If $d = d_0$ then the inequality is strict.

Proof. For $i = 0, \ldots, d$ we denote by P_i^d the polytope

$$P_i^d = \text{conv}\{0, \pm e^1, \ldots, \pm e^i, e^{i+1}, \ldots, e^d\}.$$

Thus $P_0^d = R^d$ and $P_d^d = C^d$. Now let $\mu = G(aR^d)/W(aR^d)$ and

$$d_0 = \min\{j \mid G(aP_i^j)/W(aP_i^j) \geq \mu \text{ for some } 0 \leq i \leq j\}.$$

Obviously we have $d_0 \leq d$. Further, we have from the additivity of the functionals $G(\cdot)$, $W(\cdot)$, for $1 \leq i \leq j$

$$G(aP_i^j) = 2G(aP_{i-1}^j) - G(aP_{i-1}^{j-1}),$$
$$W(aP_i^j) = 2W(aP_{i-1}^j) - W(aP_{i-1}^{j-1}).$$

By definition we have $G(aP_i^j)/W(aP_i^j) < \mu$ for all $0 \leq i \leq j < d_0$, and for some i_0 we have $G(aP_{i_0}^{d_0})/W(aP_{i_0}^{d_0}) \geq \mu$. If $i_0 = d_0$ we have $d_0 < d$ and there is nothing to prove. Otherwise

$$\frac{G(aP_{i_0+1}^{d_0})}{W(aP_{i_0+1}^{d_0})} = \frac{2G(aP_{i_0}^{d_0}) - G(aP_{i_0}^{d_0-1})}{2W(aP_{i_0}^{d_0}) - W(aP_{i_0}^{d_0-1})}$$
$$> \frac{2\mu W(aP_{i_0}^{d_0}) - \mu W(aP_{i_0}^{d_0-1})}{2W(aP_{i_0}^{d_0}) - W(aP_{i_0}^{d_0-1})} = \mu$$

It follows that $G(aP_k^{d_0})/W(aP_k^{d_0}) > \mu$ for $i_0 < k \leq d_0$, and thus we have the assertion. $\square$

4. Numerical calculations

In view of Theorem 3.1., we should expect that WILLS's conjecture fails for a dimension rather lower than 410. This is indeed the case. Let $n \in \mathbb{N}$ be a positive integer. The number of lattice points of nR^d is $G(nR^d) = \binom{n+d}{d}$, and for the regular crosspolytope C^d we have [PS, p. 4]

$$G(nC^d) = \sum_{i=0}^{d} 2^{d-i} \binom{d}{i} \binom{n}{d-i}.$$

By Theorem 2.1. and Theorem 2.2. we can compute the ratio G/W for nR^d and nC^d to any desired precision by numerical integration. The resulting values for $d = 4, 9, 16, \ldots, 484$ and $n = \sqrt{d}$ are summarized in the following table:

d	n	$G(nC^d)/W(nC^d)$	$G(nR^d)/W(nR^d)$	d	n	$G(nC^d)/W(nC^d)$	$G(nR^d)/W(nR^d)$
4	2	0.6971	0.8096	169	13	0.7766	0.4338
9	3	0.5177	0.6243	196	14	0.9303	0.4717
16	4	0.4362	0.5051	225	15	1.1314	0.5194
25	5	0.4004	0.4347	256	16	1.3957	0.5786
36	6	0.3896	0.3941	289	17	1.7445	0.6517
49	7	0.3960	0.3717	324	18	2.2080	0.7417
64	8	0.4168	0.3617	361	19	2.8278	0.8524
81	9	0.4515	0.3611	400	20	3.6625	0.9887
100	10	0.5014	0.3683	441	21	4.7946	1.1570
121	11	0.5692	0.3827	484	22	6.3412	1.3654
144	12	0.6589	0.4044				

In particular we have $G(14 \cdot C^{207})/W(14 \cdot C^{207}) = 1.0022$ and $G(16 \cdot R^{401})/W(16 \cdot R^{401}) = 1.0031$. The difference beween HADWIGER's result for R^d (dimension 410) and our result arises from the fact that HADWIGER did not compute the intrinsic volumes for R^d but for a circumscribed regular simplex [H].

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